Math. Proc. Camb. Phil. Soc. (2002), **132**, 353 © Cambridge Philosophical Society

DOI: 10.1017/S0305004101005692 *Printed in the United Kingdom*

A new path integral representation for the solutions of the Schrödinger, heat and stochastic Schrödinger equations

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(*Received* 11 *June* 1999; *revised* 3 *February* 2000)

Abstract

Solutions to the Schrödinger, heat and stochastic Schrödinger equation with rather general potentials are represented, both in x- and p-representations, as integrals over the path space with respect to σ-finite measures. In the case of x-representation, the corresponding measure is concentrated on the Cameron–Martin Hilbert space of curves with L^2-integrable derivatives. The case of the Schrödinger equation is treated by means of a regularization based on the introduction of either complex times or continuous non-demolition observations.

0. *Introduction*

The Feynman path integral is known to be a powerful tool in different domains of physics. At the same time, the mathematical theory underlying lots of (often formal) physical calculations is far from being complete. In the most usual approaches to the mathematically rigorous construction of the Feynman integral, one defines this integral as some generalised functional on an appropriate space of functions, or as the limit of certain discrete approximations (see e.g. [**ABB**, **ACH**, **AH**, **AKS1**, **CW**, **E**, **ET**, **HKPS**, **K1**, **K2**, **SS**, **T1**, **T2**, **TZ**] and references therein). An alternative way of constructing the path integral, initiated in [**MCh**], defines it as an expectation with respect to a certain compound Poisson process (see e.g. [**ChQ**, **Co1**, **Co2**, **Ga**, **M1**, **MCh**, **PQ**] and references therein). Most of these approaches can cover only a very restrictive class of potentials, namely the case of potentials being the Fourier transforms of finite measures (and some its generalizations, say with potentials depending in a certain way on velocity). In this paper we construct Feynman's integral as a genuine integral over a bona fide σ-finite positive measure on a path space for a rather general class of potentials. However, for the case of the Schrödinger equation the integral is not absolutely convergent (usually) and needs a certain regularization, which are of the same kind as one usually uses to give a rigorous meaning to a conditionally convergent finite-dimensional Riemann integral.

Furthermore, in the original papers of Feynman the path integral was defined (heuristically) in such a way that the solutions to the Schrödinger equation were expressed as the integrals of the function $\exp\{iS\}$, where S is the classical action along the paths. It seems that rigorously the corresponding measure was not constructed even for the case of the heat equation with sources (notice that in the famous Feynman-Kac formula that gives rigorous path integral representation for the solutions to the heat equation a part of the action is actually "hidden" inside the Wiener measure). An attempt to construct such a measure leads to a different kind of path integral, which is discussed in the last section of the paper, together with its representation in the Fock space. The latter allows us to represent this integral as an integral over the Wiener measure.

0·1. *Potentials being Fourier transforms of finite measures*

The starting point for the present study is the representation to the solutions of the Schrödinger equation with a potential being the Fourier transform of a finite measure in terms of the expectation of a certain functional over the path space of a certain compound Poisson process. As was mentioned, the main ingredient of this representation was first given in [**MCh**]. We shall start here with a simple proof of this representation, which clearly indicates the road for the generalizations that are the subject of this paper.

Let the function $V = V_{\mu,f}$ have the form

$$V(x) = V_{\mu,f}(x) = \int_{\mathcal{R}^d} e^{ipx} f(p) M(dp), \tag{0·1}$$

where f is a bounded measurable complex-valued function and M is a positive finite Borel measure on $\mathcal{R}^d$ with $\lambda_M = M(\mathcal{R}^d) < \infty$. In order to represent Feynman's integral in probabilistic terms, it is convenient to assume that M has no atom at the origin, i.e. $M(\{0\}) = 0$. This assumption is by no means restrictive, because one can ensure its validity by shifting V by an appropriate constant. Under this assumption, if

$$W(x) = \int_{\mathcal{R}^d} e^{ipx} M(dp), \tag{0·2}$$

then the equation

$$\frac{\partial u}{\partial t} = \left(W\left(\frac{1}{i}\frac{\partial}{\partial y}\right) - \lambda_M \right) u, \tag{0·3}$$

or equivalently

$$\frac{\partial u}{\partial t} = \int (u(y+\xi) - u(y)) M(d\xi), \tag{0·4}$$

defines a Feller semigroup, which is the semigroup associated with the compound Poisson process having the Lévy measure M (see e.g. [**J**] or [**Pr**] for the necessary background in the theory of Lévy processes) (notice only that the condition $M(\{0\}) = 0$ ensures that M is actually a measure on $\mathcal{R}^d \backslash \{0\}$, i.e. it is a finite Lévy measure). As is well known, such a process has almost surely piecewise constant paths. More precisely, a random path Y of this process on the interval of time $[0,t]$ starting at a point y is defined by a finite, say n, number of the moments of jumps $0 < s_1 < \cdots < s_n \leqslant t$, which are distributed according to the Poisson process N with the intensity $\lambda_M = M(\mathcal{R}^d)$, and by the independent jumps $\delta_1, \ldots, \delta_n$ at these moments, each of which is a random variable with values in $\mathcal{R}^d \backslash \{0\}$ and with the distribution defined by the probability measure M/λ_M. This path has the form

$$Y_y(s) = y + Y^{s_1\dots s_n}_{\delta_1\dots\delta_n}(s) = \begin{cases} Y_0 = y, & s < s_1, \\ Y_1 = y + \delta_1, & s_1 \leqslant s < s_2, \\ \dots & \\ Y_n = y + \delta_1 + \delta_2 + \dots + \delta_n, & s_n \leqslant s \leqslant t. \end{cases} \tag{0·5}$$

We shall denote by $E_y^{[0,t]}$ the expectation with respect to the process defined by (0.4).

Consider the Schrödinger equation

$$\frac{\partial \psi}{\partial t} = \frac{i}{2}\Delta\psi - iV(x)\psi, \tag{0·6}$$

where V is a function (possibly complex-valued) of form (0.1). The equation on the inverse Fourier transform

$$u(y) = \tilde{\psi}(y) = (2\pi)^{-d}\int_{\mathscr{R}^d} e^{-iyx}\psi(x)dx$$

of ψ (or (0.6) in momentum representation) clearly has the form

$$\frac{\partial u}{\partial t} = -\frac{i}{2}y^2 u - iV\left(\frac{1}{i}\frac{\partial}{\partial y}\right)u. \tag{0·7}$$

Proposition 0·1. *Let u_0 be a bounded continuous function. Then the solution to the Cauchy problem of* (0·7) *with the initial function u_0 has the form*

$$u(t,y) = \exp\{t\lambda_M\}E_y^{[0,t]}[F(Y(.))u_0(Y(t))], \tag{0·8}$$

where, if Y has form (0·5),

$$F(Y(.)) = \exp\left\{-\frac{i}{2}\sum_{j=0}^{n}(Y_j,Y_j)(s_{j+1}-s_j)\right\}\prod_{j=1}^{n}(-if(\delta_j)) \tag{0·9}$$

(s_{n+1} *is assumed to be equal to t in this formula*).

In particular, choosing u_0 to be the exponential function e^{iyx_0} one obtains a path integral representation for the Green function of (0·6) in momentum representation.

0·2. *Path integral as a sum of finite-dimensional integrals*

One way to visualize the integral (0·8) is by rewriting it as a sum of finite dimensional integrals. To this end, let us introduce some notations. Let $PC_p(s,t)$ (or shortly $PC_p(t)$, if $s=0$) denote the set of all right continuous and piecewise-constant paths $[s,t]\mapsto\mathscr{R}^d$ starting from the point p, and let $PC_p^n(s,t)$ denote its subset consisting of the paths with exactly n discontinuities. Topologically, PC_p^0 is a point and $PC_p^n = \mathrm{Sim}_t^n\times(\mathscr{R}^d\backslash\{0\})^n$, $n=1,2,\dots$, where

$$\mathrm{Sim}_t^n = \{s_1,\dots,s_n\colon 0 < s_1 < s_2 < \dots < s_n \leqslant t\}$$

denotes the standard n-dimensional Simplex. In fact, the numbers s_j stand for the moments of jumps, and n copies of $\mathscr{R}^d\backslash\{0\}$ stand for the values of these jumps (see (0·5)). Clearly to each σ-finite measure M on $\mathscr{R}^d\backslash\{0\}$ (or on $\mathscr{R}^d$ without an atom at the origin) corresponds the σ-finite measure $M^{PC}=M^{PC}(t,p)$ on $PC_p(t)$, which is defined as the sum of measures M_n^{PC}, $n=0,1,\dots$, with each M_n^{PC} being the measure on $PC_p^n(t)$ defined as the product-measure of the Lebesgue measure on Sim_t^n and of

n copies of the measure M on $\mathscr{R}^d$, i.e. if Y is parametrized as in (0·5), then

$$M_n^{PC}(dY(.)) = ds_1 \cdots ds_n M(d\delta_1) \cdots M(d\delta_n).$$

From the well known structure of the Poisson process it follows that (0·8) can be rewritten in the form

$$u(t,y) = \int_{PC_y(t)} M^{PC}(dY(.))F(Y(.))u_0(Y(t)), \tag{0·10}$$

or, equivalently, as

$$u(t,y) = \sum_{n=0}^{\infty} u_n(t,y) = \sum_{n=0}^{\infty} \int_{PC_y^n(t)} M_n^{PC}(dY(.))F(Y(.))u_0(Y(t)). \tag{0·11}$$

The integrals in this series can be written more explicitly (see as usual (0·5) for the parametrization of the paths Y) as

$$\begin{aligned} u_n(t,y) &= \int_{PC_y^n(t)} M_n^{PC}(dY(.))F(Y(.))u_0(Y(t)) \\ &= \int_{\mathrm{Sim}_t^n} ds_1 \cdots ds_n \int_{\mathscr{R}^d} \cdots \int_{\mathscr{R}^d} M(d\delta_1) \cdots M(d\delta_n) \\ &\quad \times F(y + Y_{\delta_1 \cdots \delta_n}^{s_1 \ldots s_n}(.))u_0(y + \delta_1 + \cdots + \delta_n). \end{aligned} \tag{0·12}$$

Notice that the multiplier $\exp\{t\lambda_M\}$ in (0·8) is due to the fact that the integral in (0·8) is not exactly over the measure M^{PC}, but over a probability measure obtained from M^{PC} by an appropriate normalization (namely, $M^{PC}(PC_p^1(t)) = t + O(t^2)$ for small t, and the normalized measure of the corresponding Poisson process is such that the probability of $PC_p^1(t)$ is $\lambda_M(t + O(t^2)) \exp\{-t\lambda_M\}$ and the jumps are distributed according to the normalized measure M/λ_M).

0·3. *Connection with perturbation theory*

A simple proof of (0·11) can be obtained from non-stationary perturbation theory, which we recall now for further references. It is well known that one can rewrite (0·6) in the integral form

$$\psi(t) = e^{i\Delta t/2}\psi_0 - i\int_0^t e^{i\Delta(t-s)/2} V\psi(s)ds \tag{0·6$'$}$$

(here V means actually the operator of multiplication on the function V), which contains not only the information from (0·6), but also the information from the initial data ψ_0. Though, strictly speaking, (0·6′) is not quite equivalent to (0·6) (because, for instance, a solution to (0·6′) may not belong to the domain of the operator Δ) under reasonable assumptions on V (for example, if V is bounded, or $V \in L^p + L^\infty$ with $p \geqslant \max(2, d/2)$, which is quite enough for our purposes) the solutions to (0·6′) defines the Schrödinger evolution $e^{it(\Delta/2 - V)}$, see e.g. [**Y**] (or earlier paper [**Ho**]), where (0·6′) is used to prove the existence of the Schrödinger propagator in even more general cases of time-dependent potentials.

Substituting the left-hand side of (0·6′) in its right-hand side and iterating this procedure one obtains for ψ the standard perturbation theory representation

$$\psi(t) = \big[e^{i\Delta t/2} - i\int_0^t e^{i\Delta(t-s)/2}Ve^{i\Delta s/2}ds + (-i)^2\int_0^t ds\int_0^s d\tau e^{i\Delta(t-s)/2}Ve^{i\Delta(s-\tau)/2}Ve^{i\Delta\tau/2} + \cdots\big]\psi_0. \qquad (0{\cdot}13)$$

Therefore, if (0·13) is convergent, then its sum defines a solution to (0·6′). Clearly, this is the case for bounded functions V, but actually holds also for more general V (see [**Y**]).

In order to see how (0·11) follows from (0·13), it is convenient to write (0·13) in p-representation and then consider it as a series in the Banach space $C_0(\mathscr{R}^d)$ of continuous functions vanishing at infinity. In p-representation the operator of multiplication on V takes the form $-iV(-i(\partial/\partial y))$ (which is just the operator of convolution with the measure μ) and the operator $e^{i\Delta t/2}$ is the multiplication on $e^{-ity^2/2}$. Thus, under the assumptions of the Proposition 0·1 one presents the solution to the Cauchy problem for (0·7) in the form

$$u(t,y) = \sum_{j=0}^{\infty} I_j(t,y) = I_0(t,y) + (\mathscr{F}I_0)(t,y) + (\mathscr{F}^2I_0)(t,y) + \cdots, \qquad (0{\cdot}14)$$

where $\mathscr{F}$ is the integral operator acting by the formula

$$(\mathscr{F}\phi)(t,y) = -i\int_0^t ds\int_{\mathscr{R}^d} M(dv-y)g(t-s,y)f(v-y)\phi(s,v) \qquad (0{\cdot}15)$$

and

$$g(t,y) = \exp\{-it(y,y)/2\}, \quad I_0 = g(t,y)u_0(y).$$

Clearly the terms of (0·14) can be obtained from the corresponding terms of (0·11) and (0·12) by a trivial linear change of the variables of integration. Consequently, if (0·14) or (0·11)–(0·12) is absolutely convergent and all its terms are absolutely convergent integrals, as is clearly the case under the assumptions of Proposition 0·1, one obtains representation (0·10) (and therefore (0·8)) for the solution $u(t,x)$.

0·4. *Regularization by introducing complex times or continuous non-demolition observation*

The first objective of this paper, which is carried out in Section 1, is to generalize Proposition 0·1, or more precisely, representation (0·11), to a wider class of potentials. In general, however, the terms of (0·14) would not be absolutely convergent integrals, or, even worse, (0·14) would not be convergent at all. To deal with this situation, one has to use some regularizations of the Schrödinger equation. In our approach, this regularization will be of the same kind as is used to define standard finite-dimensional (but not absolutely convergent) integrals. The most relevant finite-dimensional example (which motivates our approach to the corresponding infinite-dimensional integral) is the integral

$$(U_0f)(x) = (2\pi ti)^{-d/2}\int_{\mathscr{R}^d}\exp\left\{-\frac{|x-\xi|^2}{2ti}\right\}f(\xi)d\xi \qquad (0{\cdot}16)$$

defining the free propagator $e^{it\Delta/2}f$. This integral may be not well defined for a general $f \in l^2(\mathcal{R}^d)$. One of the ways to define this integral is based on the observation that according to the spectral theorem

$$e^{it\Delta/2}f = \lim_{\epsilon \to 0_+} e^{it(1-i\epsilon)\Delta/2}f \tag{0·17}$$

in $L^2(\mathcal{R}^d)$ for all $t > 0$ (i.e. one can approximate real times t by complex times $t(1 - i\epsilon)$). Since

$$(e^{it(1-i\epsilon)\Delta/2}f)(x) = (2\pi t(i+\epsilon))^{-d/2} \int_{\mathcal{R}^d} \exp\left\{-\frac{|x-\xi|^2}{2t(i+\epsilon)}\right\} f(\xi)\, d\xi$$

and the integral on the right-hand side of this equation is already absolutely convergent for all $f \subset L^2(\mathcal{R}^d)$, one can define the integral in (0·16) by the formula

$$(U_0 f)(x) = \lim_{\epsilon \to 0_+} (2\pi t(i+\epsilon))^{-d/2} \int_{\mathcal{R}^d} \exp\left\{-\frac{|x-\xi|^2}{2t(i+\epsilon)}\right\} f(\xi)\, d\xi. \tag{0·18}$$

We shall use the same approach for Feynman's integral. Namely, if the operator $-\Delta/2 + V(x)$ is self-adjoint and bounded from below, by the spectral theorem

$$\exp\{it(\Delta/2 - V(x))\}f = \lim_{\epsilon \to 0_+} \exp\{it(1-i\epsilon)(\Delta/2 - V(x))\}f, \tag{0·19}$$

where the limit is understood in the sense of the strong convergence. In other words, solutions to (0·6) can be approximated by the solutions to the regularized equation

$$\frac{\partial \psi}{\partial t} = \tfrac{1}{2}(i+\epsilon)\Delta\psi - (i+\epsilon)V(x)\psi, \tag{0·20}$$

i.e. to the Schrödinger equation in complex time. Clearly the corresponding integral equation (analogue of (0·6′)) can be obtained from (0·6′) by replacing there i to $i + \epsilon$ everywhere. It has the form

$$\psi(t) = e^{(i+\epsilon)\Delta t/2}\psi_0 - (i+\epsilon)\int_0^t e^{(i+\epsilon)\Delta(t-s)/2} V\psi(s)\, ds. \tag{0·20′}$$

If ψ satisfies (0·20), its Fourier transform u satisfies the equation

$$\frac{\partial u}{\partial t} = -\tfrac{1}{2}(i+\epsilon)y^2 u - (i+\epsilon)V\left(\frac{1}{i}\frac{\partial}{\partial y}\right)u. \tag{0·21}$$

In this paper we shall define a measure on a path space such that for any $\epsilon > 0$ and for rather general class of potentials V, the solution $\exp\{it(1-i\epsilon)(\Delta/2 - V(x))\}u_0$ to the Cauchy problem of (0·20) can be expressed as the Lebesgue (or even the Riemann) integral of some functional F_ϵ with respect to this measure, which would give a rigorous definition (analogous to (0·18)) of an improper Riemann integral corresponding to the case $\epsilon = 0$, i.e. to (0·6). Therefore, unlike the usual method of analytical continuation often used for defining Feynman's integral, where rigorous integral is defined only for purely imaginary Planck's constant h and for real h the integral is defined as the analytical continuation by rotating h through the right angle, in our approach, the (positive σ-finite) measure is rigorously defined and is the same for all complex h with a non-negative real part, and only on the boundary $Im\, h = 0$ the corresponding integral usually becomes an improper Riemann's integral.

Surely, the idea to use (0·20) as an appropriate regularization for defining Feynman's integral is not new and goes back at least to the paper [**GY**]. However, this was not carried out there, because, as it turned out (see e.g. [**RS**]), there exist no direct generalizations of the Wiener measure that could be used to define Feynman's integral for (0·20) for any real ϵ. Here we shall carry out this regularization using a measure which differs essentially from the Wiener measure.

Equation (0·20) is certainly only one of many different ways to regularize Feynman's integral (in the same way as (0·18) does not present a unique reasonable method of regularizing integral (0·16)). However, this method is one of the simplest, because the limit (0·19) follows directly from the spectral theory, and other methods may require an additional work to obtain the corresponding convergence result. As another regularization to (0·6) one can take, for instance, the equation

$$\frac{\partial\psi}{\partial t} = \tfrac{1}{2}(i+\epsilon)\Delta\psi - iV(x)\psi, \tag{0·22}$$

which means the introduction of a complex mass.

A more physically motivated regularization can be obtained from the theory of continuous quantum measurement. Though technically the work with this regularization is more difficult than with the regularization based on (0.20), we shall describe it, because, firstly, Feynman's integral representation for continuously observed quantum system is a matter of independent interest (see e.g. [**Me1**], where heuristic Feynman's integral was first applied to continuously observed quantum systems), and secondly, the idea to use the theory of continuous observation for regularization of Feynman's integral was already discussed in physical literature (see [**Me2**] or even earlier comments in [**F**]) and it is interesting to give to this idea a rigorous mathematical justification. The idea behind this approach lies in the observation that in the process of continuous non-demolition quantum measurement a spontaneous collapse of quantum states occurs (see e.g. [**Di**, **BS**, **K4**]), which gives a sort of regularization for large x (or large momenta p) divergences of Feynman's integral.

As is well known, the standard Schrödinger equation describes an isolated quantum system. In quantum theory of open systems one considers a quantum system under observation in a quantum environment (reservoir). This leads to a generalization of the Schrödinger equation, which is called stochastic Schrödinger equation (SSE), or quantum state diffusion model, or Belavkin's quantum filtering equation (it was first obtained in the general form in [**B**], in the framework of the quantum filtering theory), see e.g. discussions and reviews in [**BHH**] or [**QO**]. In the case of a non-demolition measurement of diffusion type, the SSE has the form

$$du + (iH + \tfrac{1}{2}\lambda^2 R^\star R)u\,dt = \lambda Ru\,dW, \tag{0·23}$$

where u is the unknown aposterior (non-normalized) wave function of the given continuously observed quantum system in a Hilbert space $\mathscr{H}$, the self-adjoint operator $H = H^\star$ in $\mathscr{H}$ is the Hamiltonian of a free (non-observed) quantum system, the vector-valued operator $R = (R^1, \ldots, R^d)$ in $\mathscr{H}$ stands for the observed physical values, W is the standard d-dimensional Brownian motion, and the positive constant λ stands for the precision of measurement. The simplest natural examples of (0·23) concern the case when H is the standard quantum mechanical Hamiltonian and the observed physical value R is either position or momentum of the particle. The path integral representation of the corresponding equation in the first case is given in

[**AKS1**, **AKS2**] and [**K1**], where the path integral was defined in the spirit of the approach from [**AH**]. Here we shall consider the second case, i.e. when R stands for the momentum of the particle (and therefore one models a continuous non-demolition observation of the momentum of a quantum particle) and therefore when SSE (0·23) takes the form

$$d\psi = \left(\frac{1}{2}\left(i + \frac{\lambda}{2}\right)\Delta\psi - iV(x)\psi\right) dt + \frac{1}{i}\sqrt{\frac{\lambda}{2}}\frac{\partial}{\partial x}\psi dW. \qquad (0{\cdot}24)$$

As $\lambda \to 0$, (0.24) turns to the standard Schrödinger equation (0.6). If ψ satisfies SSE (0.24), the equation on the Fourier transform $u(y)$ of ψ clearly has the form

$$du = \left(-\frac{1}{2}\left(i + \frac{\lambda}{2}\right)y^2 u - iV\left(\frac{1}{i}\frac{\partial}{\partial y}\right)u\right) dt + \sqrt{\frac{\lambda}{2}} yu\, dW. \qquad (0{\cdot}25)$$

By Ito's formula, the solution to this equation with initial function u_0 and with vanishing potential V equals $g_\lambda^W(t,y)u_0(t,y)$ with

$$g_\lambda^W(t,y) = g_\lambda^{W(t)}(t,y) = \exp\left\{-\tfrac{1}{2}(i+\lambda)y^2 t + \sqrt{\frac{\lambda}{2}} yW(t)\right\} \qquad (0{\cdot}26)$$

and therefore the analogy of (0·6′) corresponding to (0·25) has the form

$$u(t,y) = g_\lambda^{W(t)}(t,y)u_0 - i\int_0^t g_\lambda^{W(t)-W(s)}(t-s,.)V\left(-\frac{1}{i}\frac{\partial}{\partial y}\right) g_\lambda^{W(s)}(t,.)u(s,.)\, ds. \quad (0{\cdot}27)$$

The result of Proposition 0·1 can be straightforwardly generalized to the case of (0·25) and (0·27). What is more interesting, for $\lambda > 0$ representation (0·10) of the solutions in terms of path integral holds for essentially more general potentials than for the standard Schrödinger equation itself. Therefore, equation with $\lambda > 0$ can serve as a regularization for the standard Schrödinger equation with $\lambda = 0$.

0·5. *Content of the paper*

In Section 1 we are going to obtain the path integral representation for the solutions of (0·21) and (0·26) for rather general scattering potentials V, including the Coulomb potential.

The momentum representation for wave functions is known to be usually convenient for the study of interacting quantum fields. In quantum mechanics, however, one usually deals with the Schrödinger equation in x-representation. Therefore, it is desirable to write down Feynman's integral representation directly for (0·6). The rest of our paper, namely Sections 2–4, are devoted to the path integral in x-representation. Since in p-representation our measure is concentrated on the space PC of piecewise constant paths, and since, classically, trajectories $x(t)$ and momenta $p(t)$ are connected by the equation $\dot{x} = p$, one can expect that in x-representation the correspondonding measure is concentrated on the set of continuous piecewise linear paths. This measure and the corresponding Feynman's integral will be constructed in Sections 2 and 3 for (0·6) with bounded potentials and also for a class of singular potentials. It will be shown there also how to generalize these results in order to be able to include the case of harmonic oscillator.

In Section 4 we give an alternative Feynman's integral representation for the solutions of stochastic heat and Schrödinger equations, which express the solutions to these equations in the form of an integral of the exponential of the classical action on paths and which shows clearly the connection with the semiclassical approximation. This representation is valid for a wide class of smooth potentials. For conclusion, we discuss shortly a Fock space representation of our path integral, which, on the one hand, puts it in the familiar framework of quantum stochastic calculus, and on the other hand, leads to its various representations as an expectation with respect to the Wiener, Poisson or general Lévy process.

1. *Path integral for the Schrödinger equation in p-representation*

Let V have form (0·1) (in the sense of distributions) with M being the Lebesgue measure $M_{\rm Leb}$ and $f \in L^1 + L^q$, i.e. $f = f_1 + f_2$ with $f_1 \in L^1$, $f_2 \in L^q$, with q from the interval $(1, d/(d-2))$, $d > 2$. Notice that this class of potentials includes the case of Coulomb potential $V(x) = |x|^{-1}$ in $\mathscr{R}^3$, because for this case $f(y) = |y|^{-2}$. Let $M_{\rm Leb}^{PC}$ be the measure on $PC_y(t)$ constructed from the Lebesgue measure $M_{\rm Leb}$ according to the construction of Subsection 0·2 of the Introduction.

Proposition 1·1. *Under given assumptions on V there exists a (strong) solution $u(t,x)$ to the Cauchy problem of* (0·21) *and* (0·25) *with initial data u_0, which is given in terms of Feynman's integral of type* (0·10), *more precisely*

$$u(t,y) = \int_{PC_y(t)} M_{\rm Leb}^{PC}(dY(.))F(Y(.))u_0(Y(t)), \tag{1·1}$$

where, if Y is parametrized as in (0·5),

$$F(Y(.)) = F_\epsilon(Y(.)) = \exp\left\{-\tfrac{1}{2}(i+\epsilon)\sum_{j=0}^n Y_j^2(s_{j+1}-s_j)\right\}\prod_{j=1}^n(-i(1-i\epsilon)f(\delta_j)) \tag{1·2}$$

for the case of (0·21) *and*

$$\begin{aligned}F(Y(.)) &= F_\lambda^W(Y(.))\\ &= \exp\left\{-\sum_{j=0}^n\left[\frac{i+\lambda}{2}Y_j^2(s_{j+1}-s_j) - \sqrt{\frac{\lambda}{2}}Y_j(W(s_{j+1})-W(s_j))\right]\right\}\prod_{j=1}^n(-if(\delta_j))\end{aligned} \tag{1·3}$$

for the case of (0·25).

Proof. Since the proof for (0·21) and (0·25) are quite similar, let us consider only the case of (0·25). As is explained in the Introduction, it is sufficient to prove that for any bounded continuous function ϕ the integral (0·15) with $g = g_\lambda^W$ from (0·26) is absolutely convergent (almost surely), and moreover, the corresponding series (0·14) is absolutely convergent. To this end, consider the integral

$$J = \int_{\mathscr{R}^d} |f(v-y)| g_\lambda^W(t,y)\, dy.$$

Clearly, the function g_λ^W is bounded (for a.a. W) for times from any fixed interval of the positive half-line, and for small t

$$\sup_y\{|g(t,y)|\} = \exp\{W^2(t)/4t\} \leqslant \exp\{\log|\log t|/2\} = \sqrt{|\log t|}, \tag{1·4}$$

due to the well known log log law for Brownian motion W. Hence, by assumptions on f and due to the Hölder inequality

$$J = O(\sqrt{|\log t|}) + O(1)\|g_\lambda^W(t,.)\|_{L^p},$$

where $p^{-1}+q^{-1}=1$. Since

$$\|g_\lambda^W(t,.)\|_{L^p}^p = \left(\frac{2\pi}{p\lambda t}\right)^{d/2} \exp\left\{\frac{pW^2(t)}{4t}\right\},$$

it follows that J is bounded for t from any finite interval of the positive half-line, and $J = O(\lambda t)^{-d/2p}\sqrt{|\log t|}$ for small t. Since the condition $q < d/(d-2)$ is equivalent to the condition $p > d/2$, there exists an $\epsilon \in (0,1)$ such that $J \leqslant C((\lambda t)^{-(1-\epsilon)})$. Moreover, clearly $I_0(t,y) = g(t,y)u_0(y)$ does not exceed $Kt^{-\epsilon}$ for some constant K. We can now easily estimate the terms of series (0·14). Namely, we have

$$|\mathscr{F} I_0(t,y)| \leqslant KC\lambda^{-(1-\epsilon)} \int_0^t (t-s)^{-(1-\epsilon)} s^{-\epsilon}\, ds = KC\lambda^{-(1-\epsilon)} B(\epsilon, 1-\epsilon),$$

where B denotes the Euler β-function. Similarly,

$$|\mathscr{F}^2 I_0(t,y)| \leqslant \lambda^{-2(1-\epsilon)} B(\epsilon,1-\epsilon)KC^2 \int_0^t (t-s)^{-(1-\epsilon)} ds = B(\epsilon,1-\epsilon)B(\epsilon,1)KC^2 t^\epsilon.$$

By induction we obtain the estimate

$$|\mathscr{F}^k I_0(t,y)| \leqslant KC^k \lambda^{-k(1-\epsilon)} t^{(k-1)\epsilon} B(\epsilon,1-\epsilon)B(\epsilon,1)\cdots B(\epsilon,1+(k-2)\epsilon).$$

Using the representation of the β-function in terms of Γ-function, one gets that the terms of series (0·14) are of order $t^{k\epsilon}/\Gamma(1+k\epsilon)$, which implies the convergence of this series for all t. Since we estimate all functions by their magnitude, we proved also that all terms of series (0·14) are absolutely convergent integrals, and that this series converges absolutely.

It is well known that under the assumptions of Proposition 1·1 the operator $-\Delta/2 + V(x)$ is self-adjoint and bounded from below (see e.g. [**CFKS**]). Therefore, due to (0·19), the following result is a direct consequence of Proposition 1·1.

PROPOSITION 1·2. *Assume the assumptions of Proposition* 1·2 *hold. Then for any* $u_0 \in L^\infty \bigcap L^2$, *the solution to* (0·6) *is given by the improper Feynman's integral* (0·10), *which should be understood rigorously as*

$$u(t,y) = \lim_{\epsilon\to 0} \int_{PC_y(t)} M_{\text{Leb}}^{PC}(dY(.))F_\epsilon(Y(.))u_0(Y(t)), \tag{1·5}$$

where the limit is understood in the sense of L^2*-convergence.*

As the convergence of the solutions of (0·25) to the solutions of the ordinary Schrödinger equation (0·6) (similar to (0·19)) seems to be unknown, the use of (0·25) to obtain a regularization for Feynman's integral of (0·6) similar to (1·4) requires some additional work. This seems possible to do under the assumptions of Proposition 1·2 using the technique from [**Y**]. But we shall restrict ourselves here to the case of bounded potential, which will be used also in the next section. Notice that we prove now this result using p-representation, but it automatically implies the same fact for the Schrödinger equation in x-representation.

PROPOSITION 1·3. *Let V be a bounded measurable function. Then for any $u_0 \in L^2(\mathscr{R}^d)$ the solution u_λ^W of* (0·27) (*which obviously exists and is unique, see details in Section* 2) *tends* (*almost surely*) *in L^2-sense to the solution of this equation with $\lambda = 0$ as $\lambda \to 0$.*

Proof. Due to the semigroup property of the solutions, it is clearly enough to prove the statement for small times. Using the boundedness of all operators on the right-hand side of (0·27) and (1·4), one obtains that

$$\begin{aligned}\|u_\lambda^W - u\| &\leqslant \|g_\lambda^W(t,y)u_0 - g_0^W(t,y)u_0\| + O(t)|\log t|\|u_\lambda^W - u\| \\ &= O(t)|\log t|\|u_\lambda^W - u\| + o(\lambda),\end{aligned}$$

where $o(\lambda)$ is uniform with respect to finite times t. Since $O(t)|\log t| < 1$ for small enough t, it follows that $\|u_\lambda^W - u\| = o(\lambda)$ for small t, which proves the statement of the proposition.

COROLLARY. *If V is a bounded function, and the assumptions of Proposition* 1·1 *hold, then the solution to* (0·7) *can be presented in the form*

$$u(t,y) = \lim_{\lambda \to 0} \int_{PC_y(t)} M_{\text{Leb}}^{PC}(dY(.))F_\lambda^W(Y(.))u_0(Y(t)), \tag{1·6}$$

where the limit is again understood in the L^2-sense.

2. *Path integral for the Schrödinger equation in x-representation*

As we mentioned in the Introduction, we are going to deal here with measures that are concentrated on the set of continuous piecewise linear paths. Denote this set by CPL. Let $CPL^{x,y}(0,t)$ denote the class of paths $q: [0,t] \mapsto \mathscr{R}^d$ from CPL joining x and y in time t, i.e. such that $q(0) = x$, $q(t) = y$. By $CPL_n^{x,y}(0,t)$ we denote its subclass consisting of all paths from $CPL^{x,y}(0,t)$ that have exactly n jumps of their derivative. Obviously,

$$CPL^{x,y}(0,t) = \bigcup_{n=0}^{\infty} CPL_n^{x,y}(0,t).$$

To any σ-finite measure M on $\mathscr{R}^d$ corresponds a unique σ-finite measure M^{CPL} on $CPL^{x,y}(0,t)$, which is the sum of the measures M_n^{CPL} on $CPL_n^{x,y}(0,t)$, where M_0^{CPL} is just the unit measure on the one-point set $CPL_0^{x,y}(0,t)$ and each M_n^{CPL}, $n > 0$, is the direct product of the Lebesgue measure on the moments of jumps $0 < s_1 < \cdots < s_n < t$ of the derivatives of the paths $q(.)$ and of the n copies of the measure M on the values $q(s_j)$ of the paths at these moments. In other words, if

$$q(s) = q_{\eta_1 \cdots \eta_n}^{s_1 \cdots s_n}(s) = \eta_j + (s - s_j)\frac{\eta_{j+1} - \eta_j}{s_{j+1} - s_j}, \quad s \in [s_j, s_{j+1}] \tag{2·1}$$

(where it is assumed that $s_0 = 0, s_{n+1} = t, \eta_0 = x, \eta_{n+1} = y$) is a typical path from $CPL_n^{x,y}(0,t)$ and Φ is a functional on $CPL^{x,y}(0,t)$, then

$$\begin{aligned}\int_{CPL^{x,y}(0,t)} \Phi(q(.))M^{CPL}(dq(.)) &= \sum_{n=0}^{\infty} \int_{CPL_n^{x,y}(0,t)} \Phi(q(.))M_n^{CPL}(dq(.)) \\ &= \sum_{n=0}^{\infty} \int_{\text{Sim}_t^n} ds_1 \cdots ds_n \int_{\mathscr{R}^d} \cdots \int_{\mathscr{R}^d} M(d\eta_1) \cdots M(d\eta_n)\Phi(q(.)).\end{aligned} \tag{2·2}$$

Remark. Nothing is changed if $CPL_n^{x,y}(0,t)$ is defined as the set of paths with no more than n jumps of their derivative. In fact, the measure M_n^{CPL} of the subset $CPL_{n-1}^{x,y}(0,t) \subset CPL_n^{x,y}(0,t)$ vanishes anyway, because if there is no jump, say, at the moment s_j it means that $(\eta_j - \eta_{j-1})(s_{j+1} - s_{j-1}) = (\eta_{j+1} - \eta_{j-1})(s_j - s_{j-1})$, therefore s_j can be only one point, and the Lebesgue measure has no atoms.

To express the solutions to the Schrödinger equation in terms of path integral we shall use the following functionals on $CPL^{x,y}(0,t)$ depending on any measurable function V on $\mathscr{R}^d$:

$$\begin{aligned}\Phi_\epsilon(q(.)) &\\ &= \prod_{j=1}^{n+1}(2\pi(s_j - s_{j-1})(i+\epsilon))^{-d/2} \exp\left\{-\sum_{j=1}^{n+1}\frac{|\eta_j - \eta_{j-1}|^2}{2(i+\epsilon)(s_j - s_{j-1})}\right\}\prod_{j=1}^{n}(-(i+\epsilon)V(\eta_j)) \\ &= \prod_{j=1}^{n+1}(2\pi(s_j - s_{j-1})(i+\epsilon))^{-d/2}\prod_{j=1}^{n}(-(i+\epsilon)V(\eta_j))\exp\left\{-\frac{1}{2(i+\epsilon)}\int_0^t \dot{q}^2(s)\,ds\right\},\end{aligned} \tag{2·3}$$

and

$$\begin{aligned}\Phi_\lambda^W(q(.)) &= \prod_{j=1}^{n+1}(2\pi(s_j - s_{j-1})(i+\lambda))^{-d/2} \\ &\times \exp\left\{-\sum_{j=1}^{n+1}\frac{(\eta_j - \eta_{j-1} - i\sqrt{(\lambda/2)}(W(s_j) - W(s_{j-1}))^2}{2(i+\lambda)(s_j - s_{j-1})}\right\}\prod_{j=1}^{n}(-iV(\eta_j)).\end{aligned} \tag{2·4}$$

As in Section 1, we shall denote by M_{Leb} the Lebesgue measure on $\mathscr{R}^d$ and by M_{Leb}^{CPL} the corresponding measure on $CPL(0,t)$.

PROPOSITION 2·1. *Let V be any bounded measurable function on $\mathscr{R}^d$. Then for any $\epsilon > 0$, or any $\lambda > 0$ and* a.a. *Wiener trajectories W there exists a unique solution $G_\epsilon(t,x,x_0)$ or $G_\lambda^W(t,x,x_0)$ to the Cauchy problem of* (0·20) *or* (0·24) *respectively with the Dirac initial data $\delta(x-x_0)$, i.e. the Green function for these equations. These solutions are uniformly bounded for all (x,x_0) and t from any compact interval of the open half-line and they are expressed in terms of path integral as follows*:

$$G_\epsilon(t,x,x_0) = \int_{CPL^{x,y}(0,t)} \Phi_\epsilon(q(.))M_{\text{Leb}}^{CPL}(dq(.)), \tag{2·5}$$

$$G_\lambda^W(t,x,x_0) = \int_{CPL^{x,y}(0,t)} \Phi_\lambda^W(q(.))M_{\text{Leb}}^{CPL}(dq(.)), \tag{2·6}$$

with Φ_ϵ and Φ_λ^W from (2·3) *and* (2·4). *For any $\psi_0 \in L^2(\mathscr{R}^d)$ the solution $\psi_0(t,s)$ of the Cauchy problem for* (0·6) *with the initial data ψ_0 has the form of an improper* (*not absolutely convergent*) *path integral that can be understood rigorously as either*

$$\psi(t,x) = \lim_{\epsilon\to 0_+}\int_{CPL^{x,y}(0,t)}\int_{\mathscr{R}^d}\psi_0(y)\Phi_\epsilon(q(.))M_{\text{Leb}}^{CPL}(dq(.))dy, \tag{2·7}$$

or (almost surely) as

$$\psi(t,x) = \lim_{\lambda\to 0_+} \int_{CPL^{x,y}(0,t)} \int_{\mathscr{R}^d} \psi_0(y)\Phi^W_\lambda(q(.))M^{CPL}_{\text{Leb}}(dq(.))dy, \tag{2·8}$$

where both limits are understood in L^2-sense.

Proof. Formulas (2·7) and (2·8) follow from (2·5), (2·6), (0·19) and Proposition 1·3. The proofs of (2·5) and (2·6) are similar and we shall prove only (2·5). To this end, notice that the analogue of (0·13) for the case of (0·20) has form (0·13) with i substituted by $(i+\epsilon)$ everywhere. In particular, for the Green function one has the representation

$$G_\epsilon(t,x,x_0) = G^{\text{free}}_\epsilon(t,x,x_0) - (i+\epsilon)\int_0^t \int_{\mathscr{R}^d} G^{\text{free}}_\epsilon(t-s,x-\eta)V(\eta)G^{\text{free}}_\epsilon(s,\eta-x_0)\, d\eta ds + \cdots, \tag{2·9}$$

where G^{free}_ϵ is the Green function of the 'free' (0.20) (i.e. with $V=0$):

$$G^{\text{free}}_\epsilon(t,x-x_0) = (2\pi t(i+\epsilon))^{-d/2}\exp\left\{-\frac{(x-x_0)^2}{2(i+\epsilon)t}\right\}.$$

To prove (2·5) one needs to prove that the terms of this series are absolutely convergent integrals and the series is absolutely convergent with a bounded sum. This is more or less straightforward. Namely, to prove that the second integral in this series is absolutely convergent, we must estimate the integral

$$\begin{aligned} J(t,x) &\\ = &\int_0^t \int_{\mathscr{R}^d} |2\pi(i+\epsilon)|^{-d}((t-s)s)^{-d/2}\left|\exp\left\{-\frac{(x-\eta)^2}{2(t-s)(i+\epsilon)} - \frac{(\eta-x_0)^2}{2s(i+\epsilon)}\right\}\right| ds\, d\eta \\ = &\int_0^t \int_{\mathscr{R}^d} (2\pi\sqrt{1+\epsilon^2})^{-d}((t-s)s)^{-d/2}\exp\left\{-\frac{\epsilon}{2(1+\epsilon^2)}\left[\frac{(x-\eta)^2}{t-s} + \frac{(\eta-x_0)^2}{s}\right]\right\} ds\, d\eta. \end{aligned}$$

This is a Gaussian integral in η which is calculated explicitly by well known formulas to give

$$J = t(2\pi\epsilon t)^{-d/2}\exp\left\{-\frac{\epsilon(x-x_0)^2}{2(1+\epsilon^2)t}\right\}.$$

Using induction and similar calculations we get for the nth term of series (2.9) the estimate

$$C(2\pi t)^{-d/2}(t\epsilon^{-d/2})^k/k!$$

with some constant C. This completes the proof of Proposition 2·1.

Now we are going to show how to modify the formulas obtained in the case of harmonic oscillator. Instead of (0·6), let us consider the equation

$$\frac{\partial\psi}{\partial t} = \frac{i}{2}\Delta\psi - \frac{i}{2}x^2 - iV(x)\psi, \tag{2·10}$$

where V is again a bounded measurable function, and the corresponding equation in complex times:

$$\frac{\partial\psi}{\partial t} = (i+\epsilon)[\tfrac{1}{2}\Delta\psi - \tfrac{1}{2}x^2 - V(x)]\psi. \tag{2·11}$$

The analogue of (2·9) in this case is the following representation for the Green function G_ϵ of equation (2·11):

$$G_\epsilon(t,x,x_0) = G_\epsilon^{\text{osc}}(t,x,x_0) - (i+\epsilon)\int_0^t \int_{\mathscr{R}^d} G_\epsilon^{\text{osc}}(t-s,x-\eta)V(\eta)G_\epsilon^{\text{osc}}(s,\eta-x_0)\,d\eta\,ds + \cdots, \tag{2·12}$$

where G_ϵ^{osc} is the Green function of (2·11) with $V=0$, i.e. the Green function of a quantum oscillator (in complex times):

$$G_\epsilon^{\text{osc}}(t,x-x_0) = (2\pi\sinh(t(i+\epsilon)))^{-d/2}\exp\{(i+\epsilon)S_{i+\epsilon}(t,x,x_0)\}, \tag{2·13}$$

where the two-point function

$$S_{i+\epsilon}(t,x,x_0) = -\frac{(x^2+x_0^2)\cosh(t(i+\epsilon)) - 2xx_0}{2(i+\epsilon)\sinh(t(i+\epsilon))}$$

is the classical action

$$S_{i+\epsilon}(t,x,x_0) = \int_0^t (p(s)\dot{x}(s) - H(x(s),p(s)))\,ds$$

with the complex Hamiltonian

$$H = -(i+\epsilon)^2 p^2/2 + x^2/2 \tag{2·14}$$

along the (unique) solution $(x(s),p(s))$ of the corresponding Hamiltonian system joining x_0 and x in time t. Acting as in the proof of Proposition 2·1 one obtains similarly the following result (we consider here only the regularization by means of the introduction of a complex time, the regularization by means of continuous observations can be considered quite similarly).

PROPOSITION 2·2. *Let V be any bounded measurable function on $\mathscr{R}^d$. Then for any $\epsilon > 0$, the solution $G_\epsilon(t,x,x_0)$ to the Cauchy problem of* (2·11) *with the Dirac initial data $\delta(x-x_0)$, i.e. the Green function for this equation can be expressed in terms of path integral as follows*:

$$G_\epsilon(t,x,x_0) = \int_{CPL^{x,y}(0,t)} \Phi_\epsilon(q(.))M_{\text{Leb}}^{CPL}(dq(.)), \tag{2·15}$$

with

$$\Phi_\epsilon(q(.)) = \prod_{j=1}^{n+1}(2\pi\sinh((s_j-s_{j-1})(i+\epsilon)))^{-d/2}\prod_{j=1}^{n}(-(i+\epsilon)V(\eta_j)) \times\exp\left\{(i+\epsilon)\sum_{j=1}^{n+1}S_{i+\epsilon}(s_j-s_{j-1},\eta_j,\eta_{j-1})\right\}. \tag{2·16}$$

For any $\psi_0 \in L^2(\mathscr{R}^d)$ the solution $\psi_0(t,s)$ of the Cauchy problem for (2·10) *with the initial data ψ_0 has the form*

$$\psi(t,x) = \lim_{\epsilon\to 0_+}\int_{CPL^{x,y}(0,t)}\int_{\mathscr{R}^d}\psi_0(y)\Phi_\epsilon(q(.))M_{\text{Leb}}^{CPL}(dq(.))dy, \tag{2·17}$$

where the limit is understood in L^2-sense.

In Proposition 2·1, the integrand in the path integral has the form of the exponential of the classical action of the free particle multiplied by a certain density depending on the perturbation V, see (2·3). Similar representation can be given in the case of (2·10). However, in order to interpret the sum

$$\sum_{j=1}^{n+1} S_{i+\epsilon}(s_j - s_{j-1}, \eta_j, \eta_{j-1}) \qquad (2\cdot 18)$$

as the action along a path, one should modify the path space. Namely, the path space must be defined as the set of continuous piecewise classical complex trajectories (by classical trajectories it is meant here the projections on the x-space of the solutions of the Hamiltonian system with the complex Hamiltonian (2·14) and with real end-points). This new path space is again parametrized by the turning points $s_1 < \cdots < s_n$ and the values of the paths at these moments. Therefore, (2·15)–(2·17) remain the same, but the corresponding measure is now concentrated on the new path space, where the sum (2·18) is exactly the action along the corresponding path. Notice that as $\epsilon \to 0$, the action $S_{i+\epsilon}$ tends to the standard real action along the trajectories of the standard real harmonic oscillator described by the Hamiltonian $p^2/2 + x^2/2$.

Measures concentrated on piecewise classical paths are considered in more generality in Section 4.

3. *Singular potentials*

A study of Schrödinger equations with singular potentials was started in [**BF**]. Recently there was an increase of interest in singular potentials (see e.g. [**ABD**, **AFHKL**, **Ko**] and references therein). Let us consider them from the point of view of path integral constructed here. Notice that solutions to the Schrödinger equation with a certain class of singular potentials can be constructed by Feynman's integral defined as a generalized functional in Hida's white noise space [**HKPS**].

The one-dimensional situation turns out to be of special simplicity in our approach, because in this case no regularization is needed to express the solutions to the corresponding Schrödinger equation and its propagator in terms of path integral.

Proposition 3·1. *Let V be a bounded measure on $\mathcal{R}$. Then the solution ψ_G to equation* (0·6′) *with the initial function $\psi_0(x) = \delta(x - x_0)$* (*i.e. the propagator or the Green function of* (0·6)) *exists and is a continuous function of the form*

$$\psi_G(t,x) = (2\pi i t)^{-\frac{1}{2}} \exp\left\{-\frac{|x-x_0|^2}{2ti}\right\} + O(1) \qquad (3\cdot 1)$$

uniformly for finite times. Moreover, one has the path integral representation for ψ_G of the form

$$\psi_G(t,x) = \int_{CPL^{x,y}(0,t)} \Phi(q(.)) V^{CPL}(dq(.)), \qquad (3\cdot 2)$$

where V^{CPL} is constructed from V as M^{CPL} from M in Section 2, and

$$\Phi(q(.)) = \prod_{j=1}^{n+1} (2\pi(s_j - s_{j-1})i)^{-\frac{1}{2}} \exp\left\{-\frac{1}{2i}\int_0^t \dot{q}^2(s)\, ds\right\}.$$

Proof. Since V is a finite measure, in order to prove that the terms of the series in square brackets in (0·13), i.e. (2·9) with $\epsilon = 0$ (which expresses the Green function) are absolutely convergent, one needs to estimate the integrals

$$\int_0^t ds_1\,(2\pi(t-s_1))^{-\frac{1}{2}} \int_0^{s_1} ds_2\,(2\pi(s_1-s_2))^{-\frac{1}{2}} \cdots \int_0^{s_{n-1}} ds_n(2\pi s_n)^{-\frac{1}{2}},$$

which clearly exist (one-dimensional effect!) and are expressed explicitly by the Euler β-function. One sees directly that the corresponding series is convergent, which completes the proof.

For the finite-dimensional case one needs regularization, say (0·20) or (0·24). For simplicity we consider here only the regularization by (0·20).

PROPOSITION 3·2. *Let V be a finite measure on $\mathcal{R}^d$ with an additional property that*

$$\int_{|x-x_0|\leqslant R} V(dx) \leqslant CR^{\alpha} \tag{3·3}$$

for all x_0 and R and with some constants C and $\alpha > d-2$. Then for any $\epsilon > 0$ and any bounded initial function $\psi_0 \in L^2(\mathcal{R}^d)$ there exists a unique solution $\psi_\epsilon(t,x)$ to the Cauchy problem of (0·20′) with the initial data $\psi_0(x)$. This solution has the form

$$\psi_\epsilon(t,x) = \int_{CPL^{x,y}(0,t)} \int_{\mathcal{R}^d} \psi_0(y)\Phi_\epsilon(q(.))V^{CPL}(dq(.))dy, \tag{3·4}$$

where

$$\Phi_\epsilon = \prod_{j=1}^{n+1}(2\pi(s_j-s_{j-1})(i+\epsilon))^{-d/2}(-(\epsilon+i))^n \exp\left\{-\frac{1}{2(i+\epsilon)}\int_0^t \dot{q}^2(s)\,ds\right\}.$$

Proof. One needs to prove that the terms of the series (0·13), but with $(i+\epsilon)$ instead of i everywhere, are absolutely convergent integrals, and then to estimate the corresponding series. Starting with the first non-trivial term one needs to estimate the integral

$$\begin{aligned} J &= K\int_0^t \int_{\mathcal{R}^{2d}} |2\pi(i+\epsilon)|^{-d}((t-s)s)^{-d/2} \\ &\quad\times \left|\exp\left\{-\frac{(x-\xi)^2}{2(t-s)(i+\epsilon)} - \frac{(\xi-\eta)^2}{2s(i+\epsilon)}\right\}\right| ds\,d\eta|V|(d\xi). \\ &= K\int_0^t \int_{\mathcal{R}^{2d}} (2\pi\sqrt{1+\epsilon^2})^{-d}((t-s)s)^{-d/2} \exp\left\{-\epsilon\frac{(x-\xi)^2}{2(t-s)(1+\epsilon^2)}\right\} \\ &\quad\times \exp\left\{-\epsilon\frac{(\xi-\eta)^2}{2s(1+\epsilon^2)}\right\} ds\,|V|(d\xi)d\eta, \end{aligned}$$

where $K = \sup\{|\psi_0(\eta)|\}$. Integrating in η yields

$$J \leqslant K\int_0^t (2\pi\sqrt{1+\epsilon^2})^{-d/2}(t-s)^{-d/2} \exp\left\{-\epsilon\frac{(x-\xi)^2}{2(t-s)(1+\epsilon^2)}\right\}\epsilon^{-d/2}\,ds\,|V|(d\xi).$$

Let us decompose this integral into the sum $J_1 + J_2$ of the integrals over the domains D_1 and D_2 with

$$D_1 = \left\{\xi \colon |x-\xi| \leqslant (t-s)^{-\delta+\frac{1}{2}}\right\}$$

and D_2 being its complement. Choosing $\delta > 0$ such that $\alpha(-\delta + 1/2) - d/2 > -1$ (which is possible due to the assumption on α) we get from (3·3) that

$$J_1 \leqslant KC \int_0^t (2\pi\sqrt{1+\epsilon^2})^{-d/2}(t-s)^{\alpha(-\delta+\frac{1}{2})-d/2}\, ds$$
$$= KC(1+\alpha(-\delta+\tfrac{1}{2})-d/2)^{-1}(2\pi\sqrt{1+\epsilon^2})^{-d/2}t^{1+\alpha(-\delta+\frac{1}{2})-d/2}.$$

In D_2 the integrand is a uniformly exponentially small function, and therefore using the boundedness of the measure $|V|$ we obtain for J_2 even better estimate as for J_1. Other terms are again estimated by induction, which gives the required result.

Due to (0·19), the following result is a consequence of Proposition 3·2.

PROPOSITION 3·3. *Assume the assumptions of Proposition* 3·2 *holds. If the operator* $-\Delta/2 + V$ *is self-adjoint and bounded from below, then one can take* $\lim_{\epsilon\to 0}$ *in* (3·4) (*as usual, in* L^2*-sense*) *to obtain the solutions to* (0·6).

The simplest examples of singular potentials satisfying the assumptions of Proposition 3·3 are given by the potentials being the finite sums of the Dirac measures of closed hypersurfaces in $\mathscr{R}^d$ (see e.g. [**AFHKL**]). Moreover, if V satisfies the assumptions of Proposition 3·2 and is a positive measure, then for small enough ϵ the operator $-\Delta + \epsilon V$ is self-adjoint and bounded below (see again [**AFHKL**]), and consequently the measure ϵV satisfies the assumptions of Proposition 3·3. At last, the assumptions of Proposition 3·4 surely include the standard scattering potentials from $L^p(\mathscr{R}^d)$ with $p > d/2$.

Let us note in conclusion that similar to the previous section, one can easily obtain analogous results for the case of potentials of the form $x^2 + V(x)$ with V satisfying assumptions of Proposition 3·2.

4. *Integral over classical paths and a Fock space representation*

In this section we answer first the following question: how to define a measure on path space in such a way that the solutions to the heat or Schrödinger equation could be expressed as the integrals of the function exp iS over this measure, where S is the classical action along the paths. We shall give a precise result for the heat and stochastic heat equations. The case of the Schrödinger and stochastic Schrödinger equations will be considered from this point of view in [**K3**], and we reduce ourselves here only to general comments.

At the end of the section, we discuss a Fock space representation of the measure M_{Leb}^{CPC}, which gives rise to a number of various representations of our path integral in terms of the expectation with respect to a (generally speaking, infinite-dimensional) Wiener process or to a more general Lévy process.

Consider the equation

$$h\frac{\partial u}{\partial t} = \frac{h^2}{2}\Delta u - V(x)u, \quad x \in \mathscr{R}^d, \quad t > 0, \tag{4·1}$$

where h is a positive parameter (a parameter of semiclassical approximation) and V is a smooth nonnegative function with bounded Hessian matrix: $\|V''(x)\| \leqslant C$ for some constant C and for all x. We are interested in the construction of the

Green function (or, in an alternative terminology, the fundamental solution) of (4·1), i.e. of its solution $u_G(t,x) = u_G(t,x,\xi,h)$ with the Dirac initial condition $u(0,x) = \delta(x-\xi)$.

Define the Hamiltonian $H(x,p) = p^2/2 - V(x)$. The corresponding Lagrange function L (being the Legendre transform of H with respect to the second variable) clearly has the form $L(x,v) = v^2/2 + V(x)$. Let $X(t,x,p)$, $P(t,x,p)$ denote the solution of the Hamiltonian system

$$\dot{x} = p, \quad \dot{p} = \frac{\partial V}{\partial x} \tag{4·2}$$

with the initial conditions $X(0,x,p) = x$, $P(0,x,p) = p$. It is known (see e.g. [**M2**] or [**KM**]) that there exists a t_0 such that $\det(\partial X/\partial p_0) \neq 0$ for all $t \leqslant t_0$, and for all $x, \xi \in \mathscr{R}^d$ there exists a unique solution of the Hamiltonian system (4·2) joining ξ and x in times t, i.e. there exists a unique $p_0(t,x,\xi)$ such that $X(t,\xi,p_0(t,x,\xi)) = x$. As it follows then from the calculus of variations, on the curve $X(s,\xi,p_0(t,x,\xi))$ the action-functional

$$I(y(.)) = \int_0^t L(y(s),\dot{y}(s))\,ds = \int_0^t (\tfrac{1}{2}\dot{y}^2(s) + V(y(s)))\,ds \tag{4·3}$$

defined for all continuous piecewise smooth curves $y(s)$ with fixed endpoints $y(0) = \xi$ and $y(t) = x$ attains its minimum $S(t,x,\xi)$ (which is called sometimes the two-point function for H). Let us introduce the Jacobian

$$J(t,x,\xi) = \det \frac{\partial X}{\partial p}(t,\xi,p_0(t,x,\xi)).$$

It is well known that the function

$$u_G^{as}(t,x,\xi,h) = (2\pi h)^{-d/2} J(t,x,\xi)^{-\frac{1}{2}} \exp\{-S(t,x,\xi)/h\}$$

describes the exponential asymptotics to the Green function u_G of (4·1). More precisely, it is proved in [**DKM**] that

$$u_G(t,x) = u_G^{as}(t,x,\xi,h)(1 + O(ht^3)),$$

where $O(ht^3)$ is uniform with respect to $t \leqslant t_0$, $x, \xi \in \mathscr{R}^d$. The proof of this result uses the Lévy method of reconstructing the exact fundamental solution from its approximation, which gives also a convergent series representation of this fundamental solution, i.e. the following result:

PROPOSITION 4·1 ([**DKM**, **KM**]). *Suppose additionally that the third- and the fourth-order derivatives of V are bounded in $\mathscr{R}^d$. Then, for $t \leqslant t_0$,*

$$u_G(t,x) = u_G^{as}(t,x,\xi,h) + \sum_{k=1}^{\infty} u_k(t,x,\xi,h) \tag{4·4}$$

with

$$u_k(t,x,\xi,h) = h^{k-1} \int_{\{0<s_1<\ldots,s_k<t\}} \int_{\mathscr{R}^{dk}} ds_1 \cdots ds_k d\eta_1 \cdots d\eta_k u_G^{as}(t-s_k,x,\eta_k)$$
$$\times F(s_k - s_{k-1},\eta_k,\eta_{k-1}) \cdots F(s_2 - s_1,\eta_2,\eta_1) F(s_1,\eta_1,\xi),$$

where

$$F(t,x,\xi) = \frac{1}{2}\frac{\Delta(J(t,x,\xi)^{-\frac{1}{2}})}{J(t,x,\xi)^{-\frac{1}{2}}} u_G^{as}(t,x,\xi,h) = O(t^2) u_G^{as}(t,x,\xi,h) \tag{4·5}$$

and $O(t^2)$ *here is uniform with respect to all arguments.*

It turns out that (4·4) can be easily rewritten in terms of a certain path integral. In fact, similarly to Section 2, let us introduce the set CPC of continuous piecewise classical paths, i.e. such continuous paths that are smooth and satisfy classical equations of motion (4·2) up to a finite number of points, where their derivatives may have discontinuities of the first kind. Let $CPC^{x,y}(0,t)$ denote the class of paths $q\colon [0,t] \mapsto \mathscr{R}^d$ from CPL joining y and x in time t, i.e. such that $q(0) = y$, $q(t) = x$. By $CPC_n^{x,y}(0,t)$ we denote its subclass consisting of all paths from $CPC^{x,y}(0,t)$ that have exactly n jumps of their derivative. Obviously, to any σ-finite measure M on $\mathscr{R}^d$ corresponds a unique σ-finite measure M^{CPC} on $CPC^{x,y}(0,t)$, which is the sum of the measures M_n^{CPC} on $CPC_n^{x,y}(0,t)$, where M_0^{CPC} is just the unit measure on the one-point set $CPC_0^{x,y}(0,t)$ and each M_n^{CPC}, $n > 0$, is the direct product of the Lebesgue measure on the moments of jumps $0 < s_1 < \cdots < s_n < t$ of the derivatives of the paths $q(.)$ and of the n copies of the measure M on the values $q(s_j)$ of the paths at these moments. In other words, if

$$q_\eta^s = q_{\eta_1\cdots\eta_n}^{s_1\cdots s_n}(s) \in CPC_n^{x,y}(0,t) \tag{4·6}$$

denote the path that takes values η_j at the moments s_j and is smooth and classical between these moments of time, and Φ is a functional on $CPL^{x,y}(0,t)$, then

$$\int_{CPC^{x,y}(0,t)} \Phi(q(.)) M^{CPC}(dq(.)) = \sum_{n=0}^{\infty} \int_{CPC_n^{x,y}(0,t)} \Phi(q(.)) M_n^{CPC}(dq(.))$$
$$= \sum_{n=0}^{\infty} \int_{\mathrm{Sim}_t^n} ds_1 \cdots ds_n \int_{\mathscr{R}^d} \cdots \int_{\mathscr{R}^d} M(d\eta_1)\cdots M(d\eta_n)\Phi(q(.)). \tag{4·7}$$

We will use this construction only for the the case of measure M being the standard Lebesgue measure M_{Leb}. Clearly, from Proposition 4.1 one obtains directly the following result.

Proposition 4·2. *Under the assumptions of Proposition* 4·1

$$u_G(t,x,y) = \int_{CPC^{x,y}(0,t)} \exp\left\{-\frac{1}{h}I(q(.))\right\} \Phi_{\mathrm{sem}}(q(.)) M_{\mathrm{Leb}}^{CPC}(dq(.)) \tag{4·8}$$

where the 'semiclassical density' Φ_{sem} *on a typical path* (4·6) *has the form*

$$\Phi_{sem}(q_\eta^s) = (2\pi h)^{-dn/2} h^n J(t-s_n,x,\eta_n)^{-\frac{1}{2}} \prod_{k=1}^{n} \Delta(J(s_k - s_{k-1},\eta_k,\eta_{k-1})^{-\frac{1}{2}}) \tag{4·9}$$

where it is assumed that $s_0 = 0$, $\eta_0 = y$.

Hence the Green function is expressed as the integral of the functional $\exp\{-(1/h)I(q(.))\}$ (exponential of the classical action) over the measure $\Phi_{\mathrm{sem}} M_{\mathrm{Leb}}^{CPC}$, which is actually a measure on the Cameron–Martin space of paths that are absolutely continuous and have their derivatives in L_2.

Formula (4·8) can be easily generalized to the case of stochastic heat equation (or Zakai's filtering equation) of the form

$$du = \left(\frac{h}{2}\Delta u - \frac{1}{h}V(x)u - \tfrac{1}{2}x^2 u\right) dt + xu\, dW, \tag{4·10}$$

where $dW = (dW^1, \dots, dW^d)$ is the stochastic differential of the standard Brownian motion in $\mathscr{R}^d$ and V is a smooth function. The semiclassical asymptotics to the solutions of this equation were constructed by different methods in [**K5**] and [**TZ**]. Representation given in [**K5**] is quite similar to representation (4·4) and it implies directly the following result.

PROPOSITION 4·3. *Let the derivatives of the second, third and fourth order of the function V be uniformly bounded in $\mathscr{R}^d$. Then for $t \leqslant t_0$ with some t_0 the* (*random*) *Green function for* (4·10) *can be expressed as a path integral in the following way*:

$$u_G^W(t,x,y) = \int_{CPC^{x,y}(0,t)} \exp\left\{-\frac{1}{h}I^W(q(.))\right\} \Phi_{\rm sem}^W(q(.)) M_{\rm Leb}^{CPC}(dq(.)), \tag{4·11}$$

where I^W denotes the action along the trajectories of the stochastic Hamiltonian system

$$\begin{cases} dx = p\,dt, \\ dp = ((\partial V/\partial x) + 2hx)\,dt - h\,dW, \end{cases} \tag{4·12}$$

and $\Phi_{\rm sem}^W$ has form (4·9) *with the random Jacobian J^W* (*i.e. the Jacobian along the trajectories of system* (4·12)) *instead of J everywhere.*

In [**K1**], in the case of analytic (at least locally, i.e. around the real plane) potentials, the semiclassical approximation, similar to (4·4), was constructed to the Green function of complex Schrödinger equations of type (0·21) or (0·24). This allows one to generalize the results of Propositions 4·1 and 4·2 to the case of stochastic Schrödinger equation and then, using as usual (0·19), to represent the solutions to the Schrödinger equation itself as a regularized path integral of the exponential of the classical action over a measure of the type $\Phi_{\rm sem} M_{\rm Leb}^{CPC}$, see details in [**K3**].

In the construction of path integral given above the measure on path spaces CPC or CPL is quite different from the Wiener measure, because it is concentrated on the set of almost everywhere differentiable paths. It turns out however that one can rewrite this integral as a certain Wiener integral using the Wiener chaos decomposition of the Wiener space and its Fock space representation. We shall show here how it works for the case of one-dimensional heat or Schrödinger equation with the point interaction.

Consider the (formal) heat equation

$$\frac{\partial u}{\partial t} = (\Delta/2 - \delta(x))u, \quad x \in \mathscr{R}, \tag{4·13}$$

which defines the bounded semigroup $\exp\{t(\Delta/2 - \delta(x))\}$. The Green function or the heat kernel u_G^δ for this equation can be calculated explicitly (see e.g. [**ABD**] and references therein). Using the obvious heat equation version of (3·2) (i.e. without i everywhere) with $V = \delta$ and then rewriting this formula as the sum of finite-dimensional integrals we obtain for this heat kernel the following representation:

$$u_G^\delta(t,x,x_0) = \sum_{n=0}^{\infty} \int_{\mathrm{Sim}_t^n} \prod_{j=1}^{n+1} \frac{1}{\sqrt{2\pi(s_j - s_{j-1})}} \exp\left\{-\frac{x^2}{2(t-s_n)} - \frac{x_0^2}{2s_1}\right\} ds_1 \cdots ds_n. \tag{4·14}$$

This is the integral over the set $\mathrm{Sim}_t = \bigcup_{n=0}^{\infty} \mathrm{Sim}_t^n$ with the Lebesgue measure on each Sim_t^n over a certain positive function $g = \{g_n(s_1,\ldots,s_n)\}_{n=0}^{\infty}$ on this set. It is well known (see e.g. [**Mey**]) that $L^2(\mathrm{Sim}_t)$ is isomorphic to the Fock space $\Gamma(L^2([0,t]))$ over $L^2([0,t])$. Moreover, the Wiener chaos decomposition theorem states that if $dW_{s_1}\cdots dW_{s_n}$ denotes the n-dimensional stochastic Wiener differential, then to each $f = \{f_n\} \in L^2(\mathrm{Sim}_t)$ corresponds the element $\tilde{f} \in L^2(\boldsymbol{\Omega}_t)$, where $\boldsymbol{\Omega}_t$ is the Wiener space of continuous real functions on $[0,t]$, by the formula

$$\tilde{f}(W) = \sum_{n=0}^{\infty} \int_{\mathrm{Sim}_t^n} f_n(s_1,\ldots,s_n)\, dW_{s_1}\cdots dW_{s_n},$$

and the mapping $f \mapsto \tilde{f}$ is an isometric isomorphism. In particular,

$$E\|\tilde{f}\|^2 = \|f\|^2_{L^2(\mathrm{Sim}_t)},$$

where E denotes the expectation with respect to the Wiener measure. Therefore, (4·14) can be rewritten as an integral over the Wiener measure. The same can be done for the case of Schrödinger equation. The difference is that the integrand in the Schrödinger case is not positive, but a complex function, and should be represented first as the sum of four positive functions with complex coefficients, and then each of the four terms can be rewritten in terms of the Wiener expectation.

To conclude, let us comment on the probabilistic interpretation of the path integral in x-representation. The measure M^{CPL} in (2·2) may well be not finite, for example M^{CPL}_{Leb} is not finite. But any Hilbert space can be represented as an L^2 over a probability space. For example, the obvious isomorphism of $L^2(\mathscr{R}, dx)$ with $L^2(\mathscr{R}, e^{-x^2/2}dx)$ is very useful in many situations. In the same way, an integral over M^{CPL}_{Leb} can be rewritten as an integral over the probability space (up to a normalization) $(e^{-x^2/2}dx)^{CPL}$. Thus one can always rewrite the integral from (2·2) as an expectation of a certain stochastic process, which can be taken to be an integral of the compound Poisson process that stands for the path integral formula for the solutions to the Schrödinger equation in momentum representation.

REFERENCES

[**ABB**] S. Albeverio, A. Boutet de Monvel-Bertier and Zd. Brzezniak. Stationary phase method in infinite dimensions by finite approximations: applications to the Schrödinger Equation. *Poten. Anal.* **4** (1995), 469–502.

[**ABD**] S. Albeverio, Z. Brzezniak and L. Dabrowski. Fundamental solution of the heat and Schrödinger equations with point interaction. *J. Funct. Anal.* **130** (1995), 220–254.

[**ACH**] S. Albeverio, Ph. Combe, R. Hoegh-Krohn, G. Rideau and R. Stora (Eds.). *Feynman path integrals.* LNP 106 (Springer, 1979).

[**AFHKL**] S. Albeverio, J.E. Fenstad, R. Hoegh-Krohn, W. Karwowski and T. Lindstrom. Schrödinger operators with potentials supported by null sets. In: *Ideas and methods in quantum and statistical physics.* Vol. 2 in Memory of R. Hoegh-Krohn, ed. S Albeverio et al. (Cambridge University Press, 1992), 63–95.

[**AH**] S. Albeverio and R.J. Hoegh-Krohn. Mathematical theory of Feynman path integrals. LNM **523** (Springer-Verlag, 1976).

[**AKS1**] S. Albeverio, V. N. Kolokoltsov and O. G. Smolyanov. Représentation des solutions de l'équation de Belavkin pour la mesure quantique par une version rigoureuse de la formule d'intégration fonctionnelle de Menski. *C. R. Acad. Sci. Paris, Sér. 1*, **323** (1996), 661–664.

[**AKS2**] S. Albeverio, V. N. Kolokoltsov and O. G. Smolyanov. Continuous quantum measurement: local and global approaches. *Rev. Math. Phys.* **9** (1997), 907–920.

[**B**] V. P. Belavkin. Nondemolition measurement, nonlinear filtering and dynamic programming of quantum stochastic processes. In: Modelling and Control of Systems. Proc. Bellman Continuous Workshop, Sophia-Antipolis, 1988. *LNCIS* **121** (1988), 245–265.

[**BF**] F. Berezin and L. Faddeev. A remark on Schrödinger equation with a singular potential. *Sov. Math. Dokl.* **2** (1961), 372–375.

[**BHH**] V. P. Belavkin, R. Hudson and R. Hirota (Eds.). Quantum communications and measurements. N.Y., Plenum Press, 1995.

[**CW**] P. Cartier and C. DeWitt-Morette. A new perspective on functional integration. *J. Math. Phys.* **36** (1995), 2237–2312.

[**ChQ**] A. M. Chebotarev and R. B. Quezada. Stochastic approach to time-dependent quantum tunnelling. *Russian J. of Math. Phys.* **4** (1998), 275–286.

[**CFKS**] H. L. Cycon, R. F. Froese, W. Kirsh and B. Simon. *Schrödinger operators with applications to quantum mechanics and global geometry* (Springer-Verlag, 1987).

[**Co1**] Ph. Combe et al. Generalised Poisson processes in quantum mechanics and field theory. *Phys. Rep.* **77** (1981), 221–233.

[**Co2**] Ph. Combe et al. Quantum dynamic time evolution as stochastic flows on phase space. *Physica* A **124** (1984), 561–574.

[**Di**] L. Diosi. Continuous quantum measurement and Ito formalism. *Phys. Let.* A **129** (1988), 419–423.

[**DKM**] S. Yu. Dobrokhotov, V. N. Kolokoltsov and V. P. Maslov. The splitting of the low lying energy levels of the Schrödinger operator and the asymptotics of the fundamental solution of the equation $hu_t = (h^2\Delta/2 - V(x))u$. *Teoret. Mat. Fizika* **87** (1991), 323–375. [Engl. transl. in *Theor. Math. Phys.*]

[**E**] K. D. Elworthy. *Stochastic differential equations on manifolds* (Cambridge University Press, 1982).

[**ET**] K. D. Elworthy and A. Truman. Feynman maps, Cameron-Martin formulae and anharmonic oscillators. *Ann. Inst. Henri Poincaré*, **41** (1984), 115–142.

[**F**] R. P. Feynman. Space-time approach to nonrelativistic quantum mechanics. *Rev. Mod. Phys.* **20** (1948), 367–387.

[**Ga**] B. Gaveau. Representation formulas of the Cauchy problem for hyperbolic systems generalising Dirac system. *J. Funct. Anal.* **58** (1984), 310–319.

[**GY**] I. M. Gelfand and A. M. Yaglom. Integration in functional spaces and applications in quantum physics. *Uspehki Mat. Nauk* **11** (1956), 77–114. [Engl. transl. in *J. Math. Phys.* **1:1** (1960), 48–69.]

[**HKPS**] T. Hida, H.-H. Kuo, J. Potthoff and L. Streit. *White noise. An infinite dimensional calculus* (Kluwer Academic Publishers, 1993).

[**Ho**] J. Howland. Stationary scattering theory for the time dependent Hamiltonians. *Math. Ann.* **207** (1974), 315–335.

[**J**] N. Jacob. *Pseudo-differential operators and Markov processes*. Akademie-Verlag, Mathematical Research, vol. 94 (Berlin, 1996).

[**K1**] V. Kolokoltsov. Complex calculus of variations, infinite-dimensional saddle-point method and Feynman integral for dissipative stochastic Schrödinger equation. Preprint 3/99, Dep. Math. Stat. and O.R. Nottingham Trent University, 1999.

[**K2**] V. Kolokoltsov. Complex measures on path space: an introduction to the Feynman integral applied to the Schrödinger equation. *Methodology and Computing in Applied Probability*, **1:3** (1999), 349–365.

[**K3**] V. Kolokoltsov. *Semiclassical analysis for diffusion and stochastic processes*. Monograph. Springer Lecture Notes Math. Series. (2000), v.1724.

[**K4**] V. Kolokoltsov. Localization and analytic properties of the solutions of the simplest quantum filtering equation. *Rev. Math. Phys.* **10** (1998), 801–828.

[**K5**] V. N. Kolokoltsov. Stochastic Hamilton-Jacobi-Bellman equation and stochastic WKB method. Proc. Intern. Workshop 'Idempotency', held in Bristol, October 1994 (Ed. J. Gunawardena) (Cambridge University Press, 1997), 285–302.

[**KM**] V. N. Kolokoltsov and V. P. Maslov. *Idempotent analysis and applications* (Kluwer Academic Publishers, 1997).

[**Ko**] V. Koshmanenko. *Singular quadratic forms in perturbation theory* (Kluwer Academic, 1999).

[**M1**] V. P. Maslov. *Complex Markov chains and functional Feynman integral* (Moscow, Nauka, 1976) [in Russian].

[**M2**] V. P. Maslov. Perturbation theory and asymptotical methods. MGU, Moscow, 1965 (in Russian). *French Transl. Dunod*, Paris, 1972.

[**MCh**] V. P. Maslov and A. M. Chebotarev. Processus à sauts et leur application dans la mécanique quantique, in [**ACH**], p. 58–72.

[**Me1**] M. B. Menski. Quantum restriction on the measurement of the parameters of motion of a macroscopic oscillator. *Sov. Phys. JETP* **50** (1979), 667–674.

[**Me2**] M. B. Menski. The difficulties in the mathematical definition of path integrals are overcome in the theory of continuous quantum measurements. *Teoret. i Matem. Fizika* **93** (1992), 264–272. [Engl. transl. in *Theor. Math. Phys.*]

[**Mey**] P. Meyer. *Quantum probability for probabilists*. Springer Lecture Notes Math. **1538** (Springer-Verlag, 1991).

[**PQ**] P. Pereyra and R. Quezada. Probabilistic representation formulae for the time evolution of quantum systems. *J. Math. Phys.* **34** (1993), 59–68.

[**Pr**] P. Protter. *Stochastic integration and differential equations*. Applications of Mathematics 21 (Springer-Verlag, 1990).

[**QO**] Quantum and Semiclassical Optics **8:1** (1996). Special issue on stochastic quantum optics.

[**RS**] M. Reed and B. Simon. *Methods of modern mathematical physics*, v.2, *Harmonic analysis*. Self-adjointness (Academic Press, 1978).

[**Sc**] L. S. Schulman. *Techniques and applications of path integration* (John Wiley and Sons, 1996).

[**SS**] O. G. Smolyanov and E. T. Shavgulidze. *Kontinualniye integrali* (Moscow University Press, Moscow, 1990) [in Russian].

[**T1**] A. Truman. The Feynman maps and the Wiener integral. *J. Math. Phys.* **19** (1978), 1742.

[**T2**] A. Truman. The polygonal path formulation of the Feynman path integral. In [**ACH**], pp. 73–102.

[**TZ**] A.Truman and Z. Zhao. The stochastic H-J equations, stochastic heat equations and Schrödinger equations. In: A. Truman, I. M. Davies, K. D. Elworthy (Eds.), *Stochastic analysis and application* (World Scientific Press, 1996), 441–464.

[**Y**] K. Yajima. Existence of solutions for Schrödinger evolution equations. *Comm. Math. Phys.* **110** (1987), 415–426.